Uniqueness of weak solutions in hyperbolic-parabolic systems with applications in poro-elasticity

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Abstract

We focus on the uniqueness of weak solutions to hyperbolic-parabolic PDE systems. Relevant challenges are motivated by a linear poro-elastic filtration system [1] of recent interest, coupling a 3D system of elasticity and a 3D free flow. The construction of finite-energy weak solutions can be done in several ways [1, 4], however, as is typical, weak solutions do not reside in the test class [5]. As such, an additional argument is needed for uniqueness. The challenge in obtaining a traditional energy estimate for weak solutions comes through the hyperbolic component. Interface coupling brings about regularity loss in the dynamics, as well as presenting ill-defined traces. These challenges can be circumvented via semigroup methods, but those only provide well-posedness for strong and mild solutions.

To obtain energy estimates for arbitrary weak solutions—and thus continuous dependence and uniqueness—we present two different approaches, encompassing several parameter regimes for a Biot-Stokes model [2]. For a certain degenerate case, we utilize a "hyperbolic regularization" [5], adapting a classical wave equation argument [7] in more general spaces that account for boundary coupling. In the non-degenerate (semigroup) cases, we connect finite energy weak solutions to a particular adjoint notion of weak solutions appearing in [3]. The latter are unique whenever there is an underlying semigroup, so we obtain uniqueness in the former sense through identification. Time permitting, we discuss recent results [6] complementing the classic result in [3] that provide a general tool for obtaining weak well-posedness.

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