On Leray's structure theorem

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Abstract

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C^\infty$, and let $0 < T \leq \infty$. In $[0,T) \times \Omega$ we consider a general weak solution of the Navier-Stokes equations

$$u_t - \Delta u + u \cdot \nabla u + \nabla p = f, \qquad \nabla \cdot u = 0, \qquad u|_{\partial\Omega} = 0, \qquad u|_{t=0} = u_0,$$

where $u_0 \in W^{1,2}_{0,\sigma}(\Omega)$ and $f = \operatorname{div} F$, $F \in C_0^\infty([0,T); C^\infty(\overline{\Omega}))$, are given data. Our main result concerns Leray's structure theorem, see [2, p. 244]. In particular, for the special case $F = 0$, $T = \infty$, and u satisfying the strong energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_{t_0}^{t} \|\nabla u\|_2^2 \, d\tau \leq \frac{1}{2}\|u(t_0)\|_2^2$$

for almost all $t_0 \in [0,T)$ and all $t \in [t_0, T)$, it is known [1, pp. 57] that there exists an open local in time regularity region $R \subseteq (0,T)$ such that $u \in C^\infty(R; C^\infty(\overline{\Omega}))$. We extend this result to several directions: Instead of $F = 0$, $T = \infty$ we allow $F \neq 0$, $0 < T \leq \infty$ as above, and we admit a general weak solution u in $[0,T) \times \Omega$ in the usual sense, without assuming the strong energy inequality.

References

[1] G. P. Galdi, An Introduction to the Navier-Stokes Initial-Boundary Value Problem, Birkhäuser Verlag 2000.

[2] J. Leray, Sur le Mouvement d'un Liquide Visqueux Emplissant l'Espace, Acta Math. 63 (1934), 103.