Stokes resolvent with traction boundary conditions

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Abstract

Let Ω denote an open, bounded set in $\mathbb{R}^3$ with connected C^2-boundary $\partial\Omega$, and put $\overline{\Omega}^c := \mathbb{R}^3 \backslash \overline{\Omega}$ (exterior domain). Let $\lambda \in \mathbb{C}\backslash(-\infty, 0]$. Consider the Stokes resolvent system

$$-\Delta u + \lambda u + \nabla \pi = F, \;\; \operatorname{div} u = 0 \quad \text{in } \overline{\Omega}^c, \tag{1}$$

under traction boundary conditions

$$\sum_{k=1}^{3} (\partial_j u_k + \partial_k u_j - \delta_{jk}\, \pi)\, n_k^{(\Omega)} = B_j \quad \text{on } \partial\Omega \quad \text{for } \; 1 \le j \le 3, \tag{2}$$

where $n^{(\Omega)}$ denotes the outward unit normal to Ω. Let $p \in (1, \infty)$. By $D^{1,p}(\overline{\Omega}^c)$ we denote the space of all functions $\sigma \in W^{1,1}_{loc}(\overline{\Omega}^c)$ such that $\nabla \sigma \in L^p(\overline{\Omega}^c)^3$. Let $\vartheta \in [0, \pi)$. We are interested in solutions $(u, \pi) \in W^{2,p}(\overline{\Omega}^c)^3 \times D^{1,p}(\overline{\Omega}^c)$ of (1), (2), as well as in the estimate

$$|\lambda|\, \|u\|_p + \|u\|_{2,p} + \|\nabla \pi\|_{1,p} \le \mathfrak{C}\, \|F\|_p \tag{3}$$

for $F \in L^p(\overline{\Omega}^c)^3$ and $\lambda \in \mathbb{C}$ with $|\arg \lambda| \le \vartheta,\; |\lambda| \ge \lambda_0$, where $\mathfrak{C}$ and λ_0 are constants only depending on p, ϑ and Ω (estimate uniform with respect to λ with $|\arg \lambda| \le \vartheta,\; |\lambda| \ge \lambda_0$). If inequality (3) is available, problem (1), (2) may be written as an equation in terms of an operator ("Stokes operator") generating an analytic semigroup in a suitable space. This equation, which does not involve the pressure, provides an access to the time-dependent Stokes system supplemented by boundary conditions (2).

Grubb [1] expressed a solution of (1), (2) in terms of a pseudo-differential operator on $\partial\Omega$, but did not make explicit any regularity property or estimate of this solution. Shibata, Shimizu [3], [4] and Shibata [2] addressed problem (1), (2) by reducing it to a boundary value problem in half-space in $\mathbb{R}^3$. They obtained solutions in $W^{2,p}(\overline{\Omega}^c)^3 \times D^{1,p}(\overline{\Omega}^c)$ and proved (3).

Following suggestions by T. Hishida, we showed that problem (1), (2) admits two distinct solution classes in $W^{2,p}(\overline{\Omega}^c)^3 \times D^{1,p}(\overline{\Omega}^c)$, one consisting of functions (u, π) with $\int_{\partial\Omega} u \cdot n^{(\Omega)}\, do_x = 0$ (zero flux of the velocity through $\partial\Omega$), the other one characterized by the relation $\pi|B_R^c \in L^r(B_R^c)$ for some $r \in (1, \infty)$ and some $R > 0$ with $\overline{\Omega} \subset B_R$ (L^r-integrability of the pressure near infinity). The second class exists only if $p > 3/2$, and in the case $p \geq 3$, estimate (3) holds for solutions in this class only if $\operatorname{div} F = 0$ in the sense of distributions. A Stokes operator is associated with both this classes, in the case of the second under the assumption $p > 3/2$, but without the restriction $p < 3$.

There is a one-dimensional subspace of $W^{2,p}(\overline{\Omega}^c)^3 \times D^{1,p}(\overline{\Omega}^c)$ whose nonvanishing elements (u, π) satisfy (1) with $F = 0$ and (2) with $B = 0$, with u not being constant.

Keywords: Stokes resolvent, traction boundary conditions, Stokes operator.

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